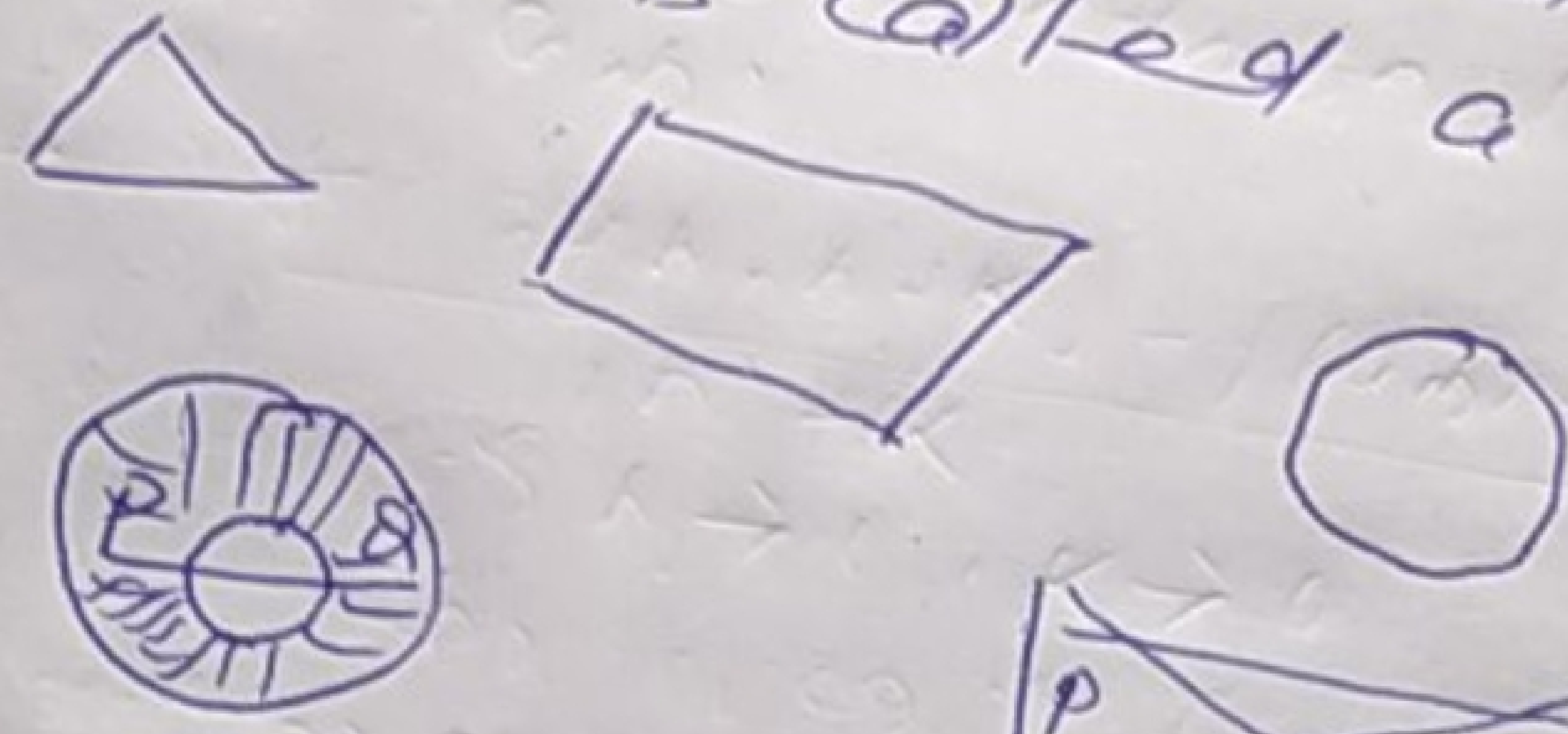
B.Sc. part-I, paper-I

(LPP) Convex set and their properties

(1) **Convex set**

Def:- Let $S \subset R^n$
If for every two points $x_1, x_2 \in S$ the line segment joining x_1 and x_2 is contained on the set S then S is called a convex set.



convex set

non convex set

Theorem A hyperplane is a convex set

proof:- Let $CX = K$ be a hyperplane and let x_1, x_2 be any points in it

we want to show that
$C(\lambda x_1 + (1-\lambda)x_2) = K,\ 0 \leq \lambda \leq 1$
since x_1, x_2 the hyperplane $CX = K$ we have
$Cx_1 = K$ and $Cx_2 = K$
Now $C(\lambda x_1 + (1-\lambda)x_2) = C(\lambda x_1) + C((1-\lambda)x_2)$

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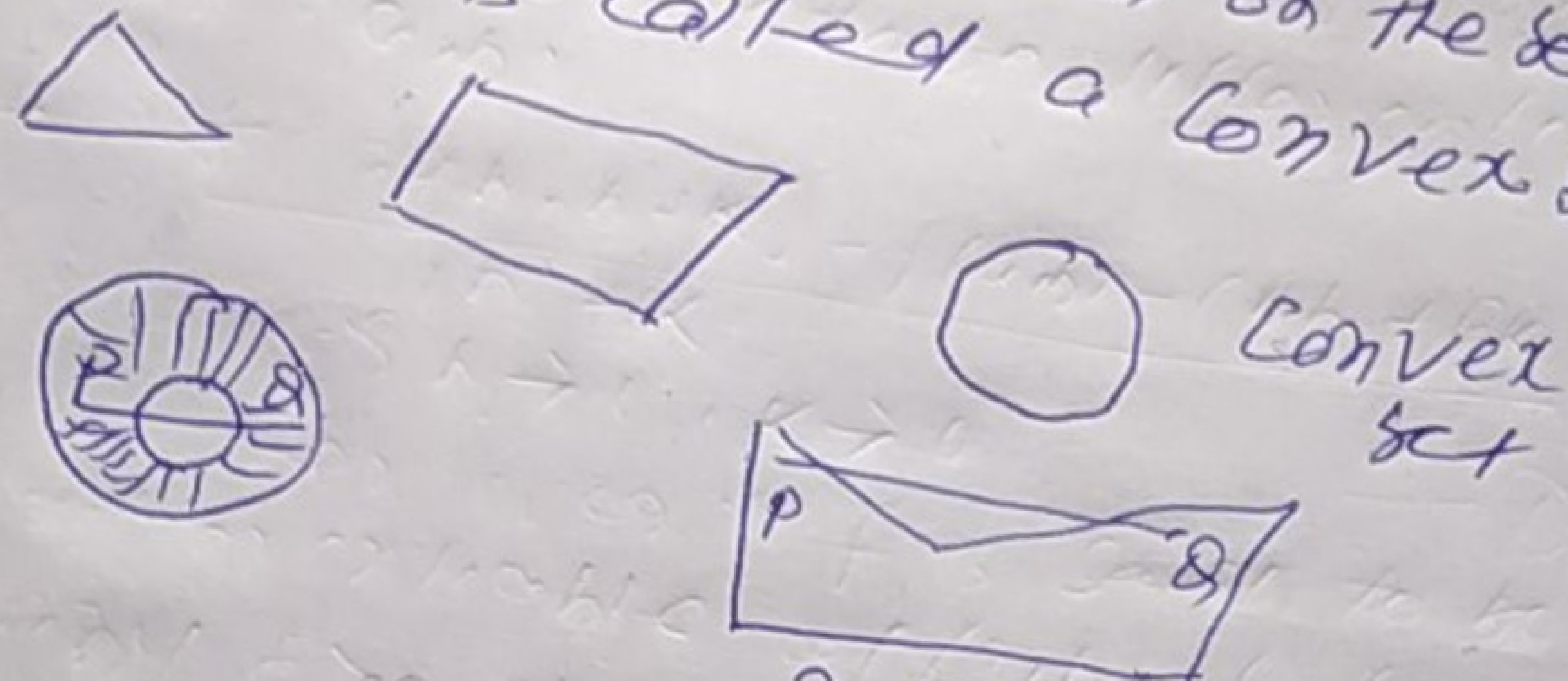

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we have
$Cx_1 = K$ and $Cx_2 = K$

Now $C[\lambda x_1 + (1-\lambda)x_2] = C(\lambda x_1) + C[(1-\lambda)x_2]$

(2) $= \lambda(Cx_1) + (1-\lambda)Cx_2$.
$= \lambda K + (1-\lambda)K = K$

Therefore the point $\lambda x_1 + (1-\lambda)x_2$ where $0 \le \lambda \le 1$ lines in the hyper plane

Hence the hyperplane is convex set

Theorem The intersection of two convex set is also a convex set.

proof:- Let A and B be two convex set and let $X = A \cap B$

it is requaried to prove that X is a convex set.

Let $x_1, x_2 \in X$ and let $S = \{x \mid x = \lambda x_1 + (1-\lambda_2)x_2, 0 \le \lambda \le 1\}$

Now $x_1, x_2 \in X \Rightarrow x_1, x_2 \in A$
$\Rightarrow S \subset A$ ($\because A$ is convex)

Again $x_1, x_2 \in X \Rightarrow x_1, x_2 \in B$
$\Rightarrow S \subset B$ ($\because B$ is convex)

$x_1, x_2 \in X \Rightarrow S \subset A$ and $S \subset B$
$\Rightarrow S \subset A \cap B$
$\Rightarrow S \subset X$

Hence X is convex set.

<u>theorem</u> :- The set of all convex combinations of a finite number of linear independent vectors $v_1, v_2, v_3 \dots v_m$ is a convex set

proof :- Let $S = \{ v \mid v = \sum_1^m \lambda_i v_i, \lambda_i \geq 0, \sum_{i=1}^m \lambda_i = 1 \}$

we have to prove that S is a convex set. Let v' and $v'' \in S$

Thus to prove that S is a convex set we need to prove that

$$\lambda v' + (1-\lambda) v'' \in S$$

Now since $v' \in S$ therefor

$v' = \sum_{i=1}^m \lambda_i' v_i$ where $\lambda_i' \geq 0$ and $\sum_{i=1}^m \lambda_i' = 1$ — (1)

Again since $v'' \in S$ therefore

$v'' = \sum_{i=1}^m \lambda_i'' v_i$ where $\lambda_i'' \geq 0$ and $\sum_{i=1}^m \lambda_i'' = 1$ — (2)

Now $v = \lambda v' + (1-\lambda) v''$; $0 \leq \lambda \leq 1$

$$= \lambda \sum_{i=1}^m \lambda_i' v_i + (1-\lambda) \sum_{i=1}^m \lambda_i'' v_i$$

from (1) and (2)

$$= \sum_{i=1}^m \{ \lambda \lambda_i' + (1-\lambda) \lambda_i'' \} v_i$$

$$= \sum_{i=1}^m \alpha_i v_i$$

since $\alpha_i = \lambda \lambda_i' + (1-\lambda) \lambda_i''$; $1, 2, 3 \dots m$

Since $0 \leq \lambda \leq 1$, $x_i' \geq 0$, $x_i'' \geq 0$; $\alpha_i \geq 0$ for each i

Also, $\sum_{i=1}^{m} \alpha_i = \sum_{i=1}^{m} \{\lambda x_i' + (1-\lambda) x_i''\}$

$= \lambda \sum_{i=1}^{m} x_i' + (1-\lambda) \sum_{i=1}^{m} x_i''$

$= \lambda \cdot 1 + (1-\lambda) \cdot 1 = 1$

Hence v is a convex combination of the vector $v_1, v_2, \ldots v_m$ i.e $v \in S$

Thus for each pair of point v', v'' is S, the line-segment joining them is contained in the set

Hence S is a convex set.

Theorem The set of all feasible solutions of a L.P. problem contitutes a convex set.

proof:- Let F be the set of all feasible solutions of the system

$$A_x = b : x \geq 0$$

we need to show that every convex combination of any two feasible solutions is also a feasible solution.

If the set F of solutions has only one element then F is a convex set. Hence the theorem is true in this case.

Now let us assume that $x^{(1)}$ and $x^{(2)}$ are they two feasible solution in F so that

$$Ax^{(1)} = b ; \quad x^{(1)} \geq 0 \quad \text{— (1)}$$

and $$Ax^{(2)} = b ; \quad x^{(2)} \geq 0 \quad \text{— (2)}$$

we now consider $x^{(0)}$ as a new point of a convex combination of $x^{(1)}$ and $x^{(2)}$

This implies that

$$x^{(0)} = \lambda x^{(1)} + (1-\lambda) x^{(2)}, \quad 0 \leq \lambda \leq 1 \quad \text{— (3)}$$

Now if we can show that $x^{(0)}$ also belongs to F, then F become convex. Thus in order to show this we must show that $x^{(0)}$ satisfies the system of constraint.

$$Ax = b ; \quad x \geq 0 \quad \text{— (4)}$$

Now
$$\begin{aligned} Ax^{(0)} &= A\left[\lambda x^{(1)} + (1-\lambda) x^{(2)}\right] \\ &= \lambda A x^{(1)} + (1-\lambda) A x^{(2)} \\ &= \lambda b + (1-\lambda) b \quad \text{from (1) and (2)} \\ &= b \end{aligned}$$

this ~~satisfied~~ shows that $x^{(0)}$ satisfies the system of constrained (4)

Also since $0 \leq \lambda \leq 1$ and since $x^{(1)}$ and $x^{(2)} \geq 0$ it follow from (3) then $x^{(0)} \geq 0$

Thus $x^{(0)} \in F$ and consequently F is convex set